Homework #8

(Physics 230A, Spring 2006)

Due 10:10 AM, May 31, 2006 (before the Wed. class)

1. (10 pts) Prove that $[\mathcal{O}^i(x), \mathcal{O}^j(y)] = 0$ if $(x-y)^2 < 0$, where the operators are of the form $\mathcal{O}^i(x) = \overline{\psi}(x)\Gamma^i\psi(x) = \overline{\psi}_\alpha(x)\Gamma^i_{\alpha\beta}\psi_\beta(x)$ with Γ^i being any 4×4 matrix.

2. (10 pts) For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x), \quad \psi^\dagger(x) \to \psi^{\dagger\prime}(x) = \psi^\dagger(x)e^{-i\alpha\gamma_5}, \tag{1}$$

where α is an arbitrary real parameter, are called chiral phase transformations.
(a) Show that the Lagrangian density $\overline{\psi}(x)(i\ \partial\!\!\!/ - m)\psi(x)$ is invariant under chiral phase transformations in the zero mass limit ($m=0$) only, and that the corresponding conserved current in this limit is the axial vector current $j^\mu_A(x) = \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)$.
(b) Define

$$\psi_L(x) \equiv \frac{1}{2}(1-\gamma^5)\psi(x), \quad \psi_R(x) \equiv \frac{1}{2}(1+\gamma^5)\psi(x). \tag{2}$$

Deduce the equations of motion for ψ_L and ψ_R for non-vanishing mass first and then shown that they decouple in the limit $m=0$.
(c) Use the result in (b) show that the Lagrangian density

$$\mathcal{L}(x) = \overline{\psi}_L(x)i\ \partial\!\!\!/\psi_L(x) \tag{3}$$

describes a zero-mass fermion with negative helicity only and a zero-mass anti-fermion with positive helicity only (i.e., it contains no fermion with positive helicity and no anti-fermion with negative helicity).
Hint: Remember that $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and γ^5 anticommutes with all γ^μ's. For a *massless* spinor $w_r(\vec{p})$ ($w_r = u_r$ or v_r), γ^5 and the helicity operator $\Sigma_{\vec{p}}$ are equivalent, $\gamma^5 w_r(\vec{p}) = \Sigma_{\vec{p}}w_r(\vec{p})$. This can be shown as follows. The massless Dirac equation implies that

$$p\!\!\!/ w_r(\vec{p}) = \gamma^0 E_{\vec{p}}w_r(\vec{p}) - p^k\gamma^k w_r(\vec{p}) = 0, \tag{4}$$

where $k=1,2,3$ and is summed over. Using $E_{\vec{p}} = |\vec{p}|$ for massless particles and multiplying the equation by $\gamma^5\gamma^0$ from the left, we have

$$|\vec{p}|\gamma^5 w_r(\vec{p}) = p^k\gamma^5\gamma^0\gamma^k w_r(\vec{p}). \tag{5}$$

Using the definition $\epsilon_{ijk}\Sigma^k = \sigma^{ij} = \frac{i}{2}[\gamma^i,\gamma^j]$, one can show that $\gamma^5\gamma^0\gamma^k = \Sigma^k$. (For example, $\gamma^5\gamma^0\gamma^1 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1 = -i(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1 = -i(\gamma^1)^2\gamma^2\gamma^3 = +i\gamma^2\gamma^3 = \frac{i}{2}[\gamma^2,\gamma^3] = \Sigma^1$ and one check the other cases.) Substituting this into the above equation gives

$$|\vec{p}|\gamma^5 w_r(\vec{p}) = \vec{\Sigma}\cdot\vec{p}\, w_r(\vec{p}), \tag{6}$$

which after dividing by $|\vec{p}|$ becomes

$$\gamma^5 w_r(\vec{p}) = (\vec{\Sigma}\cdot\hat{p})w_r(\vec{p}) = \Sigma_{\vec{p}}w_r(\vec{p}). \tag{7}$$