Homework #6

(Physics 230A, Spring 2006)

Due 10:10 AM, May 17, 2006 (before the Wed. class)

1. (10 pts) The complex Klein-Gordon fields $\phi(x)$ and $\phi^\dagger(x)$ can be expressed in terms of two independent real Klein-Gordon fields $\phi_1(x)$ and $\phi_2(x)$ by

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), \tag{1}$$

and the real fields $\phi_r(x)\,(r = 1, 2)$ are expanded as

$$\phi_r(x) = \sum_{\vec{k}} \left(\frac{1}{2V\omega_{\vec{k}}}\right)^{1/2} \left[a_r(\vec{k})e^{-ik\cdot x} + a_r^\dagger(\vec{k})e^{ik\cdot x}\right]. \tag{2}$$

(a) Show that

$$a(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) + ia_2(\vec{k})\right], \quad b(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) - ia_2(\vec{k})\right], \tag{3}$$

and derive the commutation relations for the $a, a^\dagger, b, b^\dagger$ operators from the commutation relations for $a_1, a_1^\dagger, a_2, a_2^\dagger$.

(b) Starting from the commutation relations for the real fields $\phi_{r=1,2}$ and their conjugate momenta, derive the commutation relations for ϕ, $\phi^\dagger$ and their conjugate momenta, without using their operator decompositions (i.e., do not use a and $a^\dagger$).

2. (10 pts)

(a) For a complex scalar field, derive the expression for the charge operator Q,

$$Q = q\sum_{\vec{k}} \left[N_a(\vec{k}) - N_b(\vec{k})\right], \tag{4}$$

starting from its expression in terms of fields,

$$Q = -iq\int d^3x : \left[\dot{\phi}^\dagger(x)\phi(x) - \dot{\phi}(x)\phi^\dagger(x)\right] : . \tag{5}$$

Note the normal ordering instruction. Please show what you get without normal ordering before you give the normal ordered result (i.e., apply normal ordering at the last step).

(b) Consider two complex scalar fields with the same mass. Label them as $\phi_a(x)$, where $a = 1, 2$. Using Noether's theorem, show that there are now four conserved charges, one given by the generalization of Q, and the other three given by

$$Q^i = \frac{i}{2}\int d^3x \left(\phi_a^\dagger(\sigma^i)_{ab}\pi_b^\dagger - \pi_a(\sigma^i)_{ab}\phi_b\right), \tag{6}$$

where the σ^i's are the Pauli matrices. Show that these three charges have the same commutation relations as the angular momentum (i.e., $[Q^i, Q^j] = i\epsilon^{ijk}Q^k$, the Lie algebra of the generators of SU(2)). Note that this SU(2) is not the actual spin, but some internal symmetry of the system. However, since they have the same form, it helps if you remember how a two-component spinor transforms under the rotation from quantum mechanics.

3. (10 pts) Compute a different Green's function or propagator for the scalar field theory defined by

$$\Delta_{new}(x) = \frac{1}{(2\pi)^4} \int_{C_{new}} d^4k \frac{e^{-ik\cdot x}}{k^2 - \mu^2} \tag{7}$$

where C_{new} is a contour that passes above both the $-\omega_{\vec{k}}$ and the $+\omega_{\vec{k}}$ poles. Give an expression for Δ_{new} in terms of Δ^+ and Δ^-.