Homework #5

(Physics 230A, Spring 2006)

Due 10:10 AM, May 10, 2006 (before the Wed. class)

Use $\hbar = c = 1$ for simplicity.

1. (10 pts) Mandl-Shaw Problem 2.4
Use the commutation relations

$$[\phi_r(\vec{x},t), \pi_s(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')\delta_{rs}, \quad [\phi_r(\vec{x},t), \phi_s(\vec{x}',t)] = [\pi_r(\vec{x},t), \pi_s(\vec{x}',t)] = 0 \tag{1}$$

to show that the momentum operator of the fields

$$P^j = \int d^3x\, \pi_r(x) \frac{\partial \phi_r(x)}{\partial x_j} \tag{2}$$

satisfies the equations

$$[P^j, \phi_r(x)] = -i\frac{\partial \phi_r(x)}{\partial x_j}, \quad [P^j, \pi_r(x)] = -i\frac{\partial \pi_r(x)}{\partial x_j}. \tag{3}$$

Hence, show that any operator $F(x) = F(\phi_r(x), \pi(x))$, which can be expanded in a power series in the field operators $\phi_r(x)$ and $\pi_r(x)$, satisfies

$$[P^j, F(x)] = -i\frac{\partial F(x)}{\partial x_j}. \tag{4}$$

Note that we can combine these equations with the Heisenberg equation of motion for the operator $F(x)$,

$$[H, F(x)] = -i\frac{\partial F(x)}{\partial x_0} \tag{5}$$

to obtain the covariant equations of motion

$$[P^\alpha, F(x)] = -i\frac{\partial F(x)}{\partial x_\alpha}, \tag{6}$$

where $P^0 = H$.

2. (10 pts) Mandl-Shaw Problem 3.1 plus extras
(a) From the expansion for the real Klein-Gordon field $\phi(x)$ in terms of the a and $a^\dagger$ operators, derive the following expression for the annihilation operator $a(\vec{k})$:

$$a(\vec{k}) = \frac{1}{(2V\omega_{\vec{k}})^{1/2}} \int d^3x\, e^{ik\cdot x}[i\dot{\phi}(x) + \omega_{\vec{k}}\phi(x)]. \tag{7}$$

Use this expression to derive the commutation relations for the creation and annihilation operators, $a^\dagger$ and a, from the commutation relations for the fields, $\phi(x)$ and $\pi(x)$.
(b) Prove that

$$H = \sum_{\vec{k}} \omega_{\vec{k}} \left[N(\vec{k}) + \frac{1}{2} \right] \quad \text{and} \quad \vec{P} = \sum_{\vec{k}} \vec{k} N(\vec{k}) \tag{8}$$

for the scalar field.