Use $\hbar = c = 1$ for simplicity.

1. (10 pts) Mandl-Shaw Problem 2.4
Use the commutation relations
\[ [\phi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}')\delta_{rs}, \quad [\phi_r(\vec{x}, t), \phi_s(\vec{x}', t)] = [\pi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = 0 \] (1)
to show that the momentum operator of the fields
\[ P^j = \int d^3x \, \pi_r(x) \frac{\partial \phi_r(x)}{\partial x_j} \] (2)
satisfies the equations
\[ [P^j, \phi_r(x)] = -i \frac{\partial \phi_r(x)}{\partial x_j}, \quad [P^j, \pi_r(x)] = -i \frac{\partial \pi_r(x)}{\partial x_j}. \] (3)
Hence, show that any operator $F(x) = F(\phi_r(x), \pi(x))$, which can be expanded in a power series in the field operators $\phi_r(x)$ and $\pi_r(x)$, satisfies
\[ [P^j, F(x)] = -i \frac{\partial F(x)}{\partial x_j}. \] (4)
Note that we can combine these equations with the Heisenberg equation of motion for the operator $F(x)$,
\[ [H, F(x)] = -i \frac{\partial F(x)}{\partial x_0} \] (5)
to obtain the covariant equations of motion
\[ [P^\alpha, F(x)] = -i \frac{\partial F(x)}{\partial x_\alpha}, \] (6)
where $P^0 = H$.

2. (10 pts) Mandl-Shaw Problem 3.1 plus extras
(a) From the expansion for the real Klein-Gordon field $\phi(x)$ in terms of the $a$ and $a^\dagger$ operators, derive the following expression for the annihilation operator $a(\vec{k})$:
\[ a(\vec{k}) = \frac{1}{(2V\omega_k)^{1/2}} \int d^3x \, e^{ik \cdot x} [i\dot{\phi}(x) + \omega_k \phi(x)]. \] (7)
Use this expression to derive the commutation relations for the creation and annihilation operators, $a^\dagger$ and $a$, from the commutation relations for the fields, $\phi(x)$ and $\pi(x)$.

(b) Prove that

$$H = \sum_k \omega_k \left[ N(\vec{k}) + \frac{1}{2} \right] \quad \text{and} \quad \vec{P} = \sum_k \vec{k}N(\vec{k})$$

for the scalar field.