# Homework #4

(Physics 230A, Spring 2006)

**Due 10:10 AM, May 3, 2006 (before the Wed. class)**

Set $\hbar = c = 1$ for simplicity.

1. (10 pts) Mandl-Shaw Problem 1.2
The Lagrangian of a particle of mass $m$ and charge $q$, moving in an electromagnetic field, is given by

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} m \dot{\vec{x}}^2 + q\vec{A} \cdot \dot{\vec{x}} - q\phi \tag{1}$$

where $\vec{A} = \vec{A}(\vec{x}, t)$ and $\phi = \phi(\vec{x}, t)$ are the vector and scalar potentials of the electromagnetic field at the position $\vec{x}$ and time $t$.
(a) Show that the momentum conjugate to $\vec{x}$ is given by

$$\vec{p} = m\dot{\vec{x}} + q\vec{A} \tag{2}$$

(note that the conjugate momentum is not the kinetic momentum $m\dot{\vec{x}}$ in general), and that Lagrange's equations reduce to the equations of motion of the particle,

$$m\frac{d}{dt}\dot{\vec{x}} = q\left[\vec{E} + \dot{\vec{x}} \times \vec{B}\right], \tag{3}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields at the instantaneous position of the charged particle.
(b) Derive the corresponding Hamiltonian:

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi, \tag{4}$$

and show that the resulting Hamilton equations again lead to Eqs. (2) and (3).

2. (10 pts) Mandl-Shaw Problem 2.2
The real Klein-Gordon field is described by the Hamiltonian density

$$\mathcal{H}(x) = \frac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2 + \mu^2\phi^2\right], \quad (x \equiv (t, \vec{x})) \tag{5}$$

Use the commutation relations

$$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}'), \quad [\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \tag{6}$$

to show that

$$[H, \phi(x)] = -i\pi(x), \quad [H, \pi(x)] = i(\mu^2 - \nabla^2)\phi(x), \tag{7}$$

where $H$ is the Hamiltonian of the field. From this result and the Heisenberg equations of motion for the operators $\phi(x)$ and $\pi(x)$, show that

$$\dot{\phi}(x) = \pi(x), \quad (\Box + \mu^2)\phi(x) = 0. \tag{8}$$