1. (10 pts) The Klein-Gordon equation in the presence of the electromagnetic field is

$$(i\hbar \partial_t - e\phi)^2 \psi = c^2 (-i\hbar \vec{\nabla} - \frac{e}{c} \vec{A})^2 \psi + m^2 c^4 \psi = 0.$$ 

Find out how the wavefunction $\psi$ should transform under the gauge transformation $\vec{A} \to \vec{A} - \vec{\nabla} \chi$, $\phi \to \phi + \frac{1}{e} \partial_t \chi$ to keep the Klein-Gordon equation invariant.

2. (10 pts) The Pauli-Dirac representation of the gamma matrices are

$$\gamma^0_{PD} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i_{PD} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \gamma^5_{PD} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

The Majorana representation of the gamma matrices are obtained by the transformation matrix $S$, $S \gamma^\mu_{PD} S^{-1} = \gamma^\mu_M$, with $S = \frac{1}{\sqrt{2}} \gamma^0_{PD} (1 + \gamma^2_{PD})$.

(a). Compute $\gamma^\mu_M$ and $\gamma^5_M$ and verify that all gamma matrices are imaginary in the Majorana representation.

(b). Using the Majorana representation, show that if a particle $\psi$ satisfying the Dirac equation has charge $q$, that is $i\gamma^\mu (\partial_\mu + i\frac{q}{\hbar} A^\mu) \psi - \frac{mc}{\hbar} \psi = 0$, then $\psi^*$ represents a particle with charge $-q$. 

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