A Brief Introduction to Relativistic Quantum Mechanics

Hsin-Chia Cheng, U.C. Davis

1 Introduction

In Physics 215AB, you learned non-relativistic quantum mechanics, e.g., Schrödinger equation,

$$\begin{aligned}
&E = \frac{\boldsymbol{p}^2}{2m} + V, \\
&E \rightarrow i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{p} \rightarrow -i\hbar\boldsymbol{\nabla}, \\
&i\hbar\frac{\partial}{\partial t}\Psi = \frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi.
\end{aligned} \tag{1}$$

Now we would like to extend quantum mechanics to the relativistic domain. The natural thing at first is to search for a relativistic single-particle wave equation to replace the Schrödinger equation. It turns out that the form of the relativistic equation depends on the spin of the particle,

spin-0	Klein-Gordon equation
spin-1/2	Dirac equation
spin-1	Proca equation
etc	

It is useful to study these one-particle equations and their solutions for certain problems. However, at certain point these one-particle relativistic quantum theory encounter fatal inconsistencies and break down. Essentially, this is because while energy is conserved in special relativity but mass is not. Particles with mass can be created and destroyed in real physical processes. For example, pair annihilation $e^+e^- \rightarrow 2\gamma$, muon decay $\mu^- \rightarrow e^-\bar{\nu}_e\nu_\mu$. They cannot be described by single-particle theory.

At that stage we are forced to abandon single-particle relativistic wave equations and go to a many-particle theory in which particles can be created and destroyed, that is, quantum field theory, which is the subject of the course.

2 Summary of Special Relativity

An **event** occurs at a single point in space-time and is defined by its coordinates x^{μ}, $\mu = 0, 1, 2, 3$,

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \tag{2}$$

in any given frame.

The **interval** between 2 events x^{μ} and $\bar{x}^{\mu}$ is called s,

$$\begin{aligned} s^2 &= c^2(t-\bar{t})^2 - (x-\bar{x})^2 - (y-\bar{y})^2 - (z-\bar{z})^2 \\ &= (x^0-\bar{x}^0)^2 - (x^1-\bar{x}^1)^2 - (x^2-\bar{x}^2)^2 - (x^3-\bar{x}^3)^2. \end{aligned} \tag{3}$$

We define the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

then we can write

$$s^2 = \sum_{\mu,\nu} g_{\mu\nu}(x^{\mu}-\bar{x}^{\mu})(x^{\nu}-\bar{x}^{\nu}) = g_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu}, \tag{5}$$

where we have used the Einstein convention: repeated indices (1 upper + 1 lower) are summed except when otherwise indicated.

Lorentz transformations

The postulates of Special Relativity tell us that the speed of light is the same in any inertial frame. s^2 is invariant under transformations from one inertial frame to any other. Such transformations are called Lorentz transformations. We will only need to discuss the homogeneous Lorentz transformations (under which the origin is not shifted) here,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}. \tag{6}$$

$$\begin{aligned} g_{\mu\nu}x^{\mu}x^{\nu} &= g_{\mu\nu}x'^{\mu}x'^{\nu} \\ &= g_{\mu\nu}\Lambda^{\mu}\ \ \rho x^{\rho}\Lambda^{\nu}{}_{\sigma}x^{\sigma} = g_{\rho\sigma}x^{\rho}x^{\sigma} \\ \Rightarrow g_{\rho\sigma} &= g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}. \end{aligned} \tag{7}$$

It's convenient to use a matrix notation,

$$x^{\mu} : \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \boldsymbol{x}. \tag{8}$$

$$
\begin{aligned}
s^2 &= \boldsymbol{x}^T \boldsymbol{g} \boldsymbol{x}, \\
\boldsymbol{x}' &= \boldsymbol{\Lambda} \boldsymbol{x} \\
\Rightarrow \boldsymbol{g} &= \boldsymbol{\Lambda}^T \boldsymbol{g} \boldsymbol{\Lambda}
\end{aligned} \tag{9}
$$

Take the determinant,

$$
\det \boldsymbol{g} = \det \boldsymbol{\Lambda}^T \det \boldsymbol{g} \det \boldsymbol{\Lambda}, \tag{10}
$$

so $\det \boldsymbol{\Lambda} = \pm 1$ ($+1$: proper Lorentz transformations, -1: improper Lorentz transformations).

Example: Rotations (proper):

$$
\begin{aligned}
x'^0 &= x^0 \\
x'^1 &= x^1 \cos\theta + x^2 \sin\theta \\
x'^2 &= -x^1 \sin\theta + x^2 \cos\theta \\
x'^3 &= x^3
\end{aligned} \tag{11}
$$

$$
\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}
$$

Example: Boosts (proper):

$$
\begin{aligned}
t' &= \gamma(t - \frac{v}{c^2} x^1) \quad \text{or } x'^0 = \gamma x^0 - \gamma\beta x^1 \\
x'^1 &= \gamma(x^1 - vt) \quad = \gamma x^1 - \gamma\beta x^0 \\
x'^2 &= x^2 \\
x'^3 &= x^3
\end{aligned} \tag{13}
$$

where

$$
\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{14}
$$

It's convenient to define a quantity **rapidity** η such that $\cosh\eta = \gamma$, $\sinh\eta = \gamma\beta$, then

$$
\boldsymbol{\Lambda} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}
$$

One can easily check that $\det \boldsymbol{\Lambda} = \cosh^2\eta - \sinh^2\eta = 1$.

Four-vectors, tensors

A contravariant vector is a set of 4 quantities which transforms like x^μ under a

Lorentz transformation,

$$V^\mu = \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}, \quad V'^\mu = \Lambda^\mu{}_\nu V^\nu. \tag{16}$$

A covariant vector is a set of 4 quantities which transforms as

$$A'_\mu = A_\nu \left(\boldsymbol{\Lambda}^{-1}\right)^\nu{}_\mu, \quad \boldsymbol{\Lambda}^{-1} = \boldsymbol{g}\boldsymbol{\Lambda}^T\boldsymbol{g}. \tag{17}$$

An upper index is called a contravariant index and a lower index is called a covariant index. Indices can be raised or lowered with the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu{}_\nu$. The scalar product of a contravariant vector and a covariant vector $V^\mu A_\mu$ is invariant under Lorentz transformations.

Examples: Energy and momentum form a contravariant 4-vector,

$$p^\mu = (\frac{E}{c}, p_x, p_y, p_z). \tag{18}$$

4- gradient,

$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \equiv \partial_\mu \tag{19}$$

is a covariant vector,

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \left(\boldsymbol{\Lambda}^{-1}\right)^\nu{}_\mu \frac{\partial}{\partial x^\nu}. \tag{20}$$

One can generalize the concept to tensors,

$$T'^{\mu'\nu'\cdots}{}_{\rho'\sigma'\cdots} = \Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu \cdots (\Lambda^{-1})^\rho{}_{\rho'} (\Lambda^{-1})^\sigma{}_{\sigma'} \cdots T^{\mu\nu\cdots}{}_{\rho\sigma\cdots}. \tag{21}$$

Maxwell's equations in Lorentz covariant from (Heaviside-Lorentz covention)

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \rho && (22)\\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 && (23)\\ \boldsymbol{\nabla}\times\boldsymbol{E} + \frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} &= 0 && (24)\\ \boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} &= \frac{1}{c}\boldsymbol{J} && (25) \end{aligned}$$

From the second equation we can define a vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \tag{26}$$

Subtituting it into the third equation, we have

$$\boldsymbol{\nabla} \times \left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \right) = 0, \tag{27}$$

then we can define a potential ϕ, such that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \tag{28}$$

Gauge invariance: $\boldsymbol{E}$, $\boldsymbol{B}$ are not changed under the following transformation,

$$\begin{aligned} \boldsymbol{A} &\rightarrow \boldsymbol{A} - \boldsymbol{\nabla}\chi \\ \pi &\rightarrow \phi + \frac{1}{c}\frac{\partial}{\partial t}\chi. \end{aligned} \tag{29}$$

$(c\rho, \boldsymbol{J})$ form a 4-vector J^μ. Charge conservation can be written in the Lorentz covariant form, $\partial_\mu J^\mu = 0$,

$(\phi, \boldsymbol{A})$ from a 4-vector A^μ $(A_\mu = (\phi, -\boldsymbol{A}))$, from which one can derive an antisymmetric electromagnetic field tensor,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \text{(note: } \partial^i = -\partial_i = -\frac{\partial}{\partial x^i},\ i = 1, 2, 3). \tag{30}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{31}$$

Maxwell's equations in the covariant form:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu \tag{32}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu F_{\lambda\nu} + \partial_\lambda F_{\nu\mu} + \partial_\nu F_{\mu\lambda} = 0 \tag{33}$$

where

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}, \tag{34}$$

ϵ^{0123} and its even permutation $= +1$, its odd permutation $= -1$.
Gauge invariance: $A^\mu \rightarrow A^\mu + \partial^\mu \chi$. One can check $F^{\mu\nu}$ is invariant under this transformation.